ON CARLEMAN-TYPE INEQUALITIES

PENG GAO

ABSTRACT. We give a weighted version of an inequality of Redheffer, which he used to treat Carleman's inequality. We then apply the result to get some new Carleman-type Inequalities.

1. Introduction

Throughout let $\mathbf{a} = (a_n)_{n\geq 1}$ be a nonnegative sequence with $\sum_{n=1}^{\infty} a_n < \infty$. Let $\Lambda_n = \sum_{i=1}^{n} \lambda_i$, $\lambda_i > 0$ and $G_n = (\prod_{i=1}^{n} a_i^{\lambda_i})^{1/\Lambda_n}$. The Carleman inequality asserts that

$$\sum_{n=1}^{\infty} (\prod_{k=1}^{n} a_k)^{\frac{1}{n}} \leq e \sum_{n=1}^{\infty} a_n.$$

We refer the reader to the survey article [6] and the references therein for an account of Carleman's inequality. Among the various generalizations of Carleman's inequality, we mention the result of E. Love, who proved for $\alpha > 0, \beta \geq 1, \lambda_i = i^{\alpha} - (i-1)^{\alpha}$,

$$\sum_{n=1}^{\infty} n^{\beta} (\prod_{i=1}^{n} a_i^{i^{\alpha}-(i-1)^{\alpha}})^{1/n^{\alpha}} \leq e^{\frac{\beta+1}{\alpha}} \sum_{n=1}^{\infty} n^{\beta} a_n, \tag{1.1}$$

and the constant $e^{\frac{\beta+1}{\alpha}}$ is best possible.

A remarkable proof of Carleman's inequality was given by R.Redheffer in [7] by developing the method of "recurrent inequalities". Another proof was given by him in [8] and his result has been generalized by H.Alzer[1] and most recently by J. Pečarić and K. Stolarsky[6], who proved for $b_n > 0$, $N \geq 1$,

$$\sum_{n=1}^{N} \Lambda_n (b_n - 1) G_n + \Lambda_N G_N \leq \sum_{n=1}^{N} \lambda_n G_n b_n^{\Lambda_n/\lambda_n}.$$

It's our goal in this paper to give another weighted version of Redheffer's treatment of Carleman's inequality and use it to get some new Carleman-type Inequalities.

2. Lemmas

Lemma 2.1. *Let $\Lambda_k = \sum_{i=1}^{k} \lambda_i$, $\lambda_i > 0$ and $G_k = (\prod_{i=1}^{k} a_i^{\lambda_i})^{1/\Lambda_k}$, then for $\mu_i > 0, n \geq 2$,*

$$G_1 + \sum_{i=2}^{n-1} (\frac{\Lambda_i \mu_i}{\lambda_i} - \frac{\Lambda_i}{\lambda_{i+1}}) G_i + \frac{\Lambda_n \mu_n}{\lambda_n} G_n \leq (1 + \frac{\Lambda_1}{\lambda_2}) a_1 + \sum_{i=2}^{n} \mu_i^{\frac{\Lambda_i}{\lambda_i}} a_i. \tag{2.1}$$

Proof. This is essentially due to R.Redheffer[7]. We note for $k \geq 2, \mu > 0, \eta > 0$,

$$\mu G_k - \eta a_k = G_{k-1}(\mu t - \eta t^{\frac{\Lambda_k}{\lambda_k}}) \leq G_{k-1}(\frac{\Lambda_{k-1}}{\lambda_k}) \eta^{\frac{-\lambda_k}{\Lambda_{k-1}}} (\frac{\mu \lambda_k}{\Lambda_k})^{\frac{\Lambda_k}{\Lambda_{k-1}}},$$

Date: July 22, 2003.
1991 *Mathematics Subject Classification.* Primary 26D15.
Key words and phrases. Carleman's inequality, recurrent inequality.

where $t^{\frac{\Lambda_k}{\lambda_k}} = a_k/G_{k-1}$(compare this with the one on page 686 of [7]). By setting $\mu_k\Lambda_k/\lambda_k = \mu, \eta_k = \eta = \mu_k^{\Lambda_k/\lambda_k}$, we get

$$\frac{\Lambda_k \mu_k}{\lambda_k} G_k - a_k \mu_k^{\frac{\Lambda_k}{\lambda_k}} \leq \frac{\Lambda_{k-1}}{\lambda_k} G_k. \tag{2.2}$$

The lemma then follows by adding (2.2) for $2 \leq k \leq n$ and $G_1 = a_1$ together. □

Lemma 2.2. *Let $f(x) \in C^3[a,b]$ and $f'''(x) \geq 0$ for $x \in [a,b]$. Then*

$$f(b) - f(a) \geq f'(\frac{a+b}{2})(b-a). \tag{2.3}$$

Proof. By Taylor's expansion,

$$\begin{aligned} f(b) &= f(\frac{a+b}{2}) + f'(\frac{a+b}{2})(b - \frac{a+b}{2}) + f''(\eta_1)(a-b)^2/4, \\ f(a) &= f(\frac{a+b}{2}) + f'(\frac{a+b}{2})(a - \frac{a+b}{2}) + f''(\eta_2)(a-b)^2/4, \end{aligned}$$

where $a < \eta_2 < (a+b)/2 < \eta_1 < b$. The lemma then follows by noticing $f'''(x) \geq 0$ for $x \in [a,b]$. □

3. The Main Results

Theorem 3.1. *Assume the same conditions in Lemma 2.1 and let $f(x)$ be a real valued function defined for $x \geq 2$ such that $f(n) = \Lambda_n/\lambda_n$ for $n \geq 2$ and $0 \leq f(x+1) - f(x) \leq 1/\alpha$ for some $\alpha > 0$. If $(1 + \frac{\Lambda_1}{\lambda_2}) \leq e^{1/\alpha}$ for the same α then*

$$\sum_{n=1}^{\infty} (\prod_{i=1}^{n} a_i^{\lambda_i})^{1/\Lambda_n} \leq e^{1/\alpha} \sum_{n=1}^{\infty} a_n. \tag{3.1}$$

Proof. It suffices to prove the theorem for any integer $n \geq 2$. Set $\mu_i = f(i+1)/f(i)$ in Lemma 2.1 we get

$$\sum_{i=1}^{n} G_i \leq \sum_{i=1}^{n-1} G_i + f(n+1)G_N \leq (1 + \frac{\Lambda_1}{\lambda_2})a_1 + \sum_{i=2}^{n} a_i (1 + \frac{f(i+1) - f(i)}{f(i)})^{f(i)} \leq e^{1/\alpha} \sum_{n=1}^{\infty} a_n,$$

by the conditions of the theorem and this completes the proof. □

Apply Theorem 3.1 to $\lambda_1 = 1, \lambda_i = \alpha^{i-1} - \alpha^{i-2}, i \geq 2$ for some $\alpha > 1$, then $f(x) = \alpha/(\alpha - 1)$ and we get

Theorem 3.2. *For $\alpha > 1$,*

$$\sum_{n=1}^{\infty} (a_1 \prod_{k=2}^{n} a_k^{\alpha^{k-1} - \alpha^{k-2}})^{1/\alpha^{n-1}} \leq (1 + \frac{1}{\alpha - 1})a_1 + \sum_{n=2}^{\infty} a_n. \tag{3.2}$$

Apply Theorem 3.1 to $\lambda_i = \alpha^i, i \geq 1$ for some $\alpha > 0$, then $f(i+1) - f(i) = \alpha^{-i}$ and we get

Theorem 3.3. *For $\alpha > 0, \sum_{n=1}^{\infty} e^{1/\alpha^n} a_n < \infty$,*

$$\sum_{n=1}^{\infty} (\prod_{k=1}^{n} a_k^{\alpha^{k-1}})^{(\alpha^n - 1)/(\alpha - 1)} \leq (1 + \frac{1}{\alpha})a_1 + \sum_{n=2}^{\infty} e^{1/\alpha^n} a_n \leq \sum_{n=1}^{\infty} e^{1/\alpha^n} a_n. \tag{3.3}$$

The λ_i's in Theorems 3.2-3.3 are of the "exponential" type and now we consider the cases where the λ_i's are of the "polynomial" type.

Theorem 3.4. *For $\alpha \geq 2$,*

$$\sum_{n=1}^{\infty}(\prod_{k=1}^{n} a_k^{k^\alpha-(k-1)^\alpha})^{1/n^\alpha} \leq e^{1/\alpha}\sum_{n=1}^{\infty} a_n. \tag{3.4}$$

Proof. Apply Theorem 3.1 with $\lambda_i = i^\alpha - (i-1)^\alpha, f(x) = x^\alpha/(x^\alpha - (x-1)^\alpha), x \geq 2$. Note for $\alpha \geq 1$,

$$1 + \frac{1}{2^\alpha - 1} \leq 1 + \frac{1}{\alpha} \leq e^{1/\alpha}.$$

And $f(i+1) - f(i) = f'(\xi), 2 \leq i < \xi < i+1$, with

$$0 < f'(\xi) = \frac{\alpha\xi^{\alpha-1}(\xi-1)^{\alpha-1}}{(\xi^\alpha - (\xi-1)^\alpha)^2} \leq \frac{1}{\alpha},$$

where the last inequality follows from Lemma 2.2 and the arithmetic-geometric inequality, since for $\alpha \geq 2$,

$$\xi^\alpha - (\xi-1)^\alpha \geq \alpha(\frac{\xi + (\xi-1)}{2})^{\alpha-1} \geq \alpha(\xi(\xi-1))^{(\alpha-1)/2}.$$

This completes the proof. $\square$

We note the theorem implies (1.1) for $\alpha \geq 2$(see page 40 in [2]), and one should also be able to improve the range of α in the theorem.

Let $[x]$ denote the largest integer not exceeding the real number x. For $x > 1, \alpha \geq 0$, we define $[x]^{\alpha-1}f(x) = \int_{1^-}^{x} t^{\alpha-1}d[t] = \lim_{\epsilon\to 0}\int_{1-\epsilon}^{x} t^{\alpha-1}d[t]$. Note for any integer $n \geq 2$, $f(n) = \sum_{i=1}^{n} i^{\alpha-1}n^{\alpha-1}$. Apply Theorem 3.1 with this $f(x)$, $\lambda_i = i^{\alpha-1}$ and note $1 + \frac{1}{2^{\alpha-1}} \leq 1 + 1/\alpha \leq e^{1/\alpha}$ for $\alpha \geq 2$ and for $\alpha = 2$, $f(n) = (n+1)/2$ for $\alpha = 3$, $f(n) = (n+1)(2n+1)/6n$; for $\alpha = 4$, $f(n) = (n+1)^2/4n$. In either case, one verifies directly $f(i+1) - f(i) \leq 1/\alpha$ which gives for $\alpha = 2, 3, 4$,

$$\sum_{n=1}^{\infty}(\prod_{i=1}^{n} a^{i^{\alpha-1}})^{1/\sum_{i=1}^{n} i^{\alpha-1}} \leq e^{\frac{1}{\alpha}}\sum_{n=1}^{\infty} a_n. \tag{3.5}$$

We don't know in this case whether $f(i+1) - f(i) \leq 1/\alpha$ holds in general. The case $i = 1$ implies it is necessary to have $\alpha \geq 2$. We note here by a result of G.Bennett and G. Jameson, we know $f(i+1)/(i+2) \leq f(i)/(i+1)$(Proposition 2, 4]). Hence $f(i+1) - f(i) \leq f(i)/(i+1) \leq (1 + 2^{\alpha-1})/(3 \cdot 2^{\alpha-1})$ for $i \geq 2$.

Now we let $p \neq 0, \frac{1}{p} + \frac{1}{q} = 1$ and let l^p be the Banach space of all complex sequences $\mathbf{a} = (a_n)_{n\geq 1}$ with norm

$$||\mathbf{a}|| := (\sum_{n=1}^{\infty} |a_n|^p)^{1/p} < \infty.$$

Corresponding to inequalities (3.4) and (3.5), we have the following

$$\sum_{n=1}^{\infty}(\frac{1}{n^\alpha}\sum_{i=1}^{n}(i^\alpha - (i-1)^\alpha)|a_i|)^p \leq (\frac{\alpha p}{\alpha p - 1})^p \sum_{n=1}^{\infty}|a_n|^p, \tag{3.6}$$

$$\sum_{n=1}^{\infty}(\frac{1}{\sum_{i=1}^{n} i^{\alpha-1}}\sum_{i=1}^{n} i^{\alpha-1}|a_i|)^p \leq (\frac{\alpha p}{\alpha p - 1})^p \sum_{n=1}^{\infty}|a_n|^p. \tag{3.7}$$

These two inequalities were announced to hold(see [2], page 40-41 and [3], page 407)whenever $\alpha > 0, p > 0, \alpha p > 1$. Replacing $|a_i|$ with $|a_i|^{1/p}$ and making $p \to \infty$ in (3.6), (3.7) gives back (3.4) and (3.5) respectively. It is thus interesting to ask whether one can apply Redheffer's method to give a proof of (3.6) and (3.7).

References

[1] H. Alzer, On Carleman's inequality, *Portugal. Math.*, **50** (1993), 331–334.

[2] G. Bennett,Factorizing the classical inequalities, *Mem. Amer. Math. Soc.*, **120** (1996), 1–130.

[3] G. Bennett, Inequalities complimentary to Hardy, *Quart. J. Math. Oxford Ser. (2)*, **49** (1998), 395–432.

[4] G.Bennett and G. Jameson, Monotonic averages of convex functions, *J. Math. Anal. Appl.*, **252** (2000), 410-430.

[5] E. Love, Inequalities related to Carleman's inequality, Inequalities (Birmingham, 1987), 135–141, Lecture Notes in Pure and Applied Mathematics vol. 129, Dekker, New York, 1991.

[6] J. Pečarić and K. Stolarsky, Carleman's inequality: history and new generalizations, *Aequationes Math.*, **61** (2001), 49–62.

[7] R.M. Redheffer, Recurrent inequalities, *Proc. London Math. Soc. (3)*, **17** (1967), 683–699.

[8] R.M. Redheffer, Easy proofs of hard inequalities, General inequalities 3 (ed. E.F.Beckenbach and W.Walter), 123-140, Birkhaäuser Verlag, Basel, 1983.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109

E-mail address: `penggao@umich.edu`